AGGREGATIVE PUBLIC GOOD GAMES

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Abstract

We exploit the aggregative structure of the public good model to provide a simple analysis of its properties. In contrast to the best response function approach, ours avoids the proliferation of dimensions as the number of players is increased, and can readily analyse games involving many heterogeneous players. We demonstrate the approach at work on the standard pure public economic model and show how it can analyse extensions of the basic model.

Key words: noncooperative games, public goods

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1 Introduction

The simplicity of the pure public good model has long made it a favourite topic for students of public economics. It continues to be a standard workhorse of public economics, appearing in every textbook. It is used routinely to demonstrate the ine¢ciency of decentralised resource allocation processes in the presence of externalities and to explore the properties of alternative mechanisms. Yet the special structure that contributes much to its simplicity is not fully exploited in many existing analyses. This paper introduces and applies an alternative way of analysing the public good model, which does e¤ectively exploit its special structure. We use it to provide a rigorous but elementary treatment of existence, uniqueness and comparative static issues without requiring the use of …xed point or other theorems in high dimensional spaces. Our approach suggests a revealing geometric representation. In addition to providing a simple analysis of the basic model we also apply our method to extensions of the model that allow for more complicated technologies.

2 The Basic Pure Public Good Game

Our starting point is the basic voluntary contribution model set out in Cornes and Sandler [11], [14] and Bergstrom, Blume and Varian [4]. This provides a tractable model in which to analyze the claim that nonmarket interdependencies, or externalities, can lead to an ine¢cient equilibrium outcome. Under standard assumptions, a unique Nash noncooperative equilibrium may be shown to exist. It is also well established that a pure public good will tend to be underprovided at the Nash noncooperative equilibrium in voluntary contributions. Finally, the model possesses interesting and, at …rst sight, surprising comparative static properties. In particular, the neutrality proposition asserts that, if all contributors face the same the unit cost of contributing to the public good, a redistribution of initial income between positive contributors will a¤ect neither the equilibrium values of the total provision level Q nor the individual players' private good consumption and utility levels. This section introduces the basic model and exploits its aggregative structure to provide a novel and simple derivation of these and other familiar properties.

2.1 Assumptions of the model

There are n players. Player i's preferences are represented by the utility function

$$u_i = u_i(y_i; Q) \tag{1}$$

where $y_i \geq 0$ is the quantity of a private good and Q the total quantity of a pure public good. We impose the following assumption on individual preferences:

A.1: Well-behaved individual preferences For all i, the function $u_i(\cdot)$ is everywhere strictly increasing and strictly quasiconcave in both arguments. It is also everywhere continuous and continuously differentiable.

Our second assumption concerns individual budget constraints:

A.2: Linear individual budget constraints Player i's budget constraint requires that

$$y_i + q_i \leq m_i \tag{2}$$

where $q_i \geq 0$ is her contribution to a pure public good and m_i is her income, which is assumed exogenous.

Since we assume the unit cost of contributing to be a constant and to be equal across players, there is no loss of generality in putting it equal to unity. Later, we extend the model and allow unit costs to vary across contributors.

We need to specify how individual contributions combine to determine the total quantity of the public good:

A.3: Additive Social Composition Function The total supply of the public good is the sum of all individual contributions:

$$Q = \sum_{j=1} q_j = q_i + Q_{-i} \tag{3}$$

where Q_{-i} is the sum of the contributions made by all players except i.

The budget constraint (2) may be written so as to incorporate the contributions of others explicitly as a component of player i's income endowment. Add the quantity Q_{-i} to both sides. This yields

$$y_i + Q \leq m_i + Q_{-i}. \qquad (4)$$

This requires that the value of the bundle consumed by i cannot exceed the value of her endowment point. This value is her "full income", $\phi_i \equiv m_i + Q_{-i}$. In addition, the player is restricted to allocations consistent with the condition that $y_i \leq m_i$, reflecting the assumption that she cannot undo the contributions of others and transform them into units of the private consumption good.

Player i chooses nonnegative values of y_i and q_i to maximise utility subject to her budget constraint and the prevailing value of Q_{-i}. To any non-negative value of Q_{-i} there corresponds a unique utility-maximising contribution level, $\hat{q}_i$. By varying Q_{-i} parametrically, we generate her best response function, $\hat{q}_i = b_i(Q_{-i})$. At a Nash equilibrium, every player's choice is a best response to the prevailing choices of all other players.

Figure 1 depicts an individual's preferences and constraint set. The values of Q_{-i} and m_i fix her endowment point E, and her constraint set is the area $OO'EF$, where the slope of EF is minus one, reflecting the assumption that the marginal rate of transformation between q_i and y_i is unity. Strict quasiconcavity of $u_i(.)$ implies a unique utility-maximising response, shown as the point of tangency T. The figure also shows the locus of tangencies traced out for a given value of m_i as Q_{-i} varies parametrically. The figure shows this locus to be everywhere upward-sloping. This reflects the following assumption that we impose on preferences:

A.4: Normality Let $\hat{y}_i(\phi_i)$ and $\hat{Q}_i(\phi_i)$ denote player i's preferred quantities of private and public good respectively as functions of full income given that her full income is and her preference and endowment point are consistent with her choosing an interior solution. Then for all ϕ_i^0 such that the preferred allocation is interior, and for all $\phi_i^1 > \phi_i^0$, we have $\hat{y}_i\left(\phi_i^1\right) > \hat{y}_i\left(\phi_i^0\right)$ and $\hat{Q}_i(\phi_i^1) > \hat{Q}_i(\phi_i^0)$. This holds for all i.

This requires that, wherever player i is at an interior solution at which y_i and q_i are both strictly positive, an increase in full income leads her to increase her demand for both y_i and Q. Normality implies that the locus of preferred points, player i's income expansion path, has positive and finite slope at all interior points.

2.2 The Replacement Function

2.2.1 The Individual Replacement Function

Taking the contributions of others as given, player i chooses non-negative values of y_i and q_i to maximise utility subject to her budget constraint. The optimal value of q_i - denoted by $\hat{q}_i$ - satisfies precisely one of the following conditions:

$$\frac{\partial u_i(\cdot)/\partial Q}{\partial u_i(\cdot)/\partial y_i} > 1 \text{ and } \hat{q}_i = m_i \tag{5}$$

$$\frac{\partial u_i(\cdot)/\partial Q}{\partial u_i(\cdot)/\partial y_i} = 1 \tag{6}$$

$$\frac{\partial u_i(\cdot)/\partial Q}{\partial u_i(\cdot)/\partial y_i} < 1 \text{ and } \hat{q}_i = 0. \tag{7}$$

The usual approach rearranges the first order conditions in the form of player i's 'best response function'. This expresses the optimising value $\hat{q}_i$ as a function of the contributions of others: $\hat{q}_i = b_i(Q_{-i})$. We take another route. We will show that conditions (5) – (7) define a single-valued function of the form $\hat{q}_i = r_i(Q)$. This we call player i's replacement function, for reasons that we shortly explain. The replacement function will be our principal tool of analysis.

We begin by establishing the following proposition:

Proposition 2.1 Consistent with any given value of $Q \geq 0$, there is at most a single value, $\hat{q}_i$, that satisfies the conditions (5) – (7).

Proof. Define player i's marginal rate of substitution as $mrs_i(q_i;Q) = \frac{\partial u_i(m_i - q_i;Q)/\partial Q}{\partial u_i(m_i - q_i;Q)/\partial Q}$. An increase in q_i alone implies a fall in y_i, because of the budget constraint. The normality assumption A.3 implies that the marginal rate of substitution falls as y_i alone falls while Q remains fixed. Therefore $mrs_i(q_i;Q)$ is a decreasing function of q_i and there is at most a single value, $\hat{q}_i$, satisfying $0 \leq \hat{q}_i \leq m_i$ at which the marginal rate of substitution equals the marginal rate of transformation of unity. If there is no such value, then either $mrs_i(q_i;Q) > 1$ for all $\hat{q}_i$ such that $0 \leq \hat{q}_i \leq Q$, in which case $\hat{q}_i = m_i$, or $mrs_i(q_i;Q) < 1$ for all $\hat{q}_i$ such that $0 \leq \hat{q}_i \leq Q$, in which case $\hat{q}_i = 0$. ■

Figure 2 graphs the function $mrs_i(q_i;Q)$ against q_i for a given value of Q. The three panels show the three situations that are consistent with the player's first-order conditions.

Now consider how the associated optimal value $\hat{q}_i$ responds as Q changes. For the moment, hold q_i fixed and consider how the value of the marginal rate of substitution responds to a change in Q. The normality assumption implies that an increase in Q, with q_i (and therefore $y_i = m_i - q_i$) held constant, reduces the value of $mrs_i(\cdot)$. Hence the graph of $mrs_i(\cdot)$ against q_i shifts downwards. The optimal value of q_i either falls or remains constant. If, initially, $0 < \hat{q}_i < m_i$, then $\hat{q}_i$ must fall in response to the increase in Q. To summarise,

Proposition 2.2 Consider any two values of total public good provision, Q^0 and Q^1, such that $Q^1 > Q^0$. Then $\hat{q}_i^1 \leq \hat{q}_i^0$, where $\hat{q}_i^k$ denotes the optimising value of q_i associated with the value Q^k. The inequality is strict if either $0 < \hat{q}_i^0 < m_i$ or $0 < \hat{q}_i^1 < m_i$.

We have yet to address the question of whether, given an arbitrary non-negative value of Q, there exists an associated optimising value of q_i. In fact, such a value is not generally defined for all non-negative values of Q. Consider the player's best response when the contributions of all other players is zero - that is, when player i is the sole contributor. Denote the total value of the public good in this situation by $\underline{Q}_i$: $\underline{Q}_i = b_i(0)$. We call $\underline{Q}_i$ player i's **standalone** level of contribution. In an economy of which player i is a member, we simply could not observe an equilibrium allocation in which the total provision of the public good falls short of $\underline{Q}_i$.

Propositions 2.1 and 2.2 define player i's replacement function $r_i(Q)$. The following proposition summarises its significant properties.

Proposition 2.3 If assumptions A.1, A.2, A.3 and A.4 hold, player i has a replacement function $r_i(Q)$ with the following properties:

1. $r_i(Q)$ is defined for all $Q \geq \underline{Q}_i$, where $\underline{Q}_i$ [player i's **standalone value**] is the level of the public good that player i would contribute if she were the sole contributor.
2. $r_i\left(\underline{Q}_i\right) = \underline{Q}_i$
3. $r_i(Q)$ is continuous.
4. $r_i(Q)$ is everywhere nonincreasing.

Remark 1 We call $r_i(Q)$ player i's replacement function for the following reason. Consider any $Q \geq \underline{Q}_i$. Then there is a unique quantity $Z \in \left[\underline{Q}_i; Q\right]$ such that, if the amount Z were subtracted from the quantity Q, the player's best response to the remaining quantity would exactly replace the quantity removed. In short, $Z = b_i(Q - Z)$.

2.2.2 The Aggregate Replacement Function

We now define the **aggregate replacement function** of the game, $R(Q)$:

Definition 1 The aggregate replacement function of the game $R(Q)$ is defined as

$$R(Q) = \sum_{j=1}^{n} r_j(Q).$$

The properties of the individual replacement functions are either preserved or else are modified in very slight and straightforward ways by the operation of addition by which $R(Q)$ is generated from them. The following proposition summarises the salient properties of $R(Q)$. All play a role in subsequent analysis.

Proposition 2.4 If assumptions A.1 - A.4 hold for all i, there is an aggregate replacement function, $R(Q) \equiv \sum_j r_j(Q)$, with the following properties:

1. $R(Q)$ is defined for all $Q \geq \max\left\{\underline{Q}_1; \underline{Q}_2; ...; \underline{Q}_n\right\}$
2. $R\left(\max\left\{\underline{Q}_1; \underline{Q}_2; ...; \underline{Q}_n\right\}\right) \geq \max\left\{\underline{Q}_1; \underline{Q}_2; ...; \underline{Q}_n\right\}$
3. $R(Q)$ is continuous.
4. $R(Q)$ is everywhere nonincreasing.

Use of $R(Q)$ suggests a simple alternative characterisation of a Nash equilibrium. A Nash equilibrium is an allocation at which every player is choosing her best response to the choices made by all other players. Clearly, the Nash equilibrium level of total provision, Q^*, must equal the sum of all best responses associated with the equilibrium allocation:

$$\hat{q}_1 + \hat{q}_2 + ... + \hat{q}_n = Q^*.$$

We have just shown that each best response may be described by that player's replacement function. At a Nash equilibrium, therefore,

$$r_1(Q^*) + r_2(Q^*) + \ldots + r_n(Q^*) = Q^*.$$

To summarise, we have established the following characterisation of Nash equilibrium:

Replacement function characterisation of Nash equilibrium A Nash equilibrium is an allocation $[r_1(Q^*); r_2(Q^*); r_3(Q^*); \ldots; r_n(Q^*); Q^*]$ such that

$$R(Q^*) \equiv \sum_{j=1} r_j(Q^*) = Q^*.$$

Remark 2 This characterisation does not require a proliferation of dimensions as the number of players increases. One simply adds together more functions, each of which is defined on an interval of the real line.

2.3 Nash Equilibrium: Existence and Uniqueness

At a Nash equilibrium, every player chooses her best response to the sum of all other players' choices. Recall that a Nash equilibrium is an allocation at which $R(Q) = Q$. Geometrically, it is a point at which the graph of $R(Q)$ intersects the 45^o ray through the origin in $(Q; R(Q))$ space.

Referring back to **Proposition 2.4**, Property 1 identifies the domain on which $R(Q)$ is defined. Property 2 locates a value of Q in that domain for which $R(Q) \gtreqless Q$. Properties 3 and 4 guarantee the existence of a unique value, Q^* at which $R(Q^*) = Q^*$. Thus, we can draw the following inference:

Proposition 2.5 There exists a unique Nash equilibrium in the pure public good game.

This proposition is illustrated by Figure 3, which depicts four individual replacement functions and the implied aggregate replacement function in a 4-player public good economy. The Nash equilibrium is the unique point of intersection between the graph of $R(Q)$ and the ray through the origin O with slope 1. Two observations are especially worth noting. First, existence and uniqueness are effectively established by a single elementary line of argument. Second, Properties 1- 4 of the aggregate replacement function are merely sufficient for the existence of a unique equilibrium. As we will indicate below, they are not necessary.

2.4 Nash Equilibrium: Comparative Static Properties

Suppose we are initially at an equilibrium. A player's replacement function may be written as $r(Q:\alpha_i)$, where α_i is a parameter whose value affects player i's replacement value. The effect of any shock on the equilibrium, represented as a change in the vector $\alpha \equiv (\alpha_1;\alpha_2;\ldots;\alpha_n)$, may be thought of as shifting the graphs of individual replacement functions, and therefore the aggregate replacement function, in a figure such as Figure 3. The equilibrium value of Q rises, remains unchanged, or falls according to whether, at its initial equilibrium value, the aggregate replacement value rises, remains unchanged, or falls. Comparative static properties of the Nash equilibrium are conveniently tackled by first considering the comparative static properties of individual replacement functions.

2.4.1 Comparative statics of a player's replacement function

The response to a change in income Consider an exogenous change in player i's money income, and suppose that player i is a positive contributor both before and after the income shock. The graphical depiction should make clear how our conclusions are modified if, at some stage, the player becomes constrained by the requirement that $q_i \geq 0$. Since we are interested in the effects of changes in income, we interpret the α_i's as players' income levels: $r_i(.) = r_i(Q;m_i)$ and $R(Q;m)$ where $m \equiv (m_1;m_2;\ldots;m_n)$. Now hold Q fixed and consider a discrete change in income Δm_i under the assumption that both before and after the change player i is a positive contributor. We are interested in the implied change in the replacement value - geometrically, we are interested in the vertical shift in the graph of $r_i(.)$ against Q. Under our assumptions, there is a one-to-one mapping between player i's preferred level of Q and her full income, ϕ_i. Thus, if player i is choosing optimally, an unchanged value of Q implies that her full income is unchanged. But if full income and Q are both unchanged, $\hat{x}_i$ must also be unchanged, since $\hat{x}_i + Q = \phi_i$. But we know that

$$\hat{x}_i = m_i - r_i(Q;m_i):$$

Suppose income is initially m_i^0 and suppose that it now changes to m_i^1 while Q is held fixed. Denote by $\Delta r_i(m_i:Q)$ the response of player i's replacement value : $\Delta r_i(m_i:Q) \equiv r_i(m_i^1:Q) - r_i(m_i^0:Q)$. Then we can infer that

$$\Delta\hat{x}_i = 0 = \Delta m_i - \Delta r_i(m_i:Q)$$

or

$$\Delta r_i(m_i . Q) = \Delta m_i. \tag{8}$$

In short,

Proposition 2.6 In the basic pure public good model, let player i be a positive contributor both before and after an exogenous change in money income of Δm_i. Then, at unchanged Q, her replacement value also changes by the amount Δm_i: $\Delta r_i(m_i . Q) = \Delta m_i$.

This result has a further implication. Suppose that contributors i and j have identical preferences but different income levels. Then $r_j(Q; m_j) - r_i(Q; m_i) = m_j - m_i$. The contribution of the player with the higher income exceeds that of the other player by an amount that precisely equals the difference in their income levels. Therefore, both enjoy the same private good consumption and the same utility level.

Proposition 2.7 If players i and j have identical preferences but possibly different incomes, and if they both make positive contributions at equilibrium, then both will enjoy the same utility level.

The response to a change in Q We already know that an increase in Q by itself implies a fall in a player's replacement value if that value is initially strictly positive. However, it is useful to obtain a more precise statement of this response and to relate it to more familiar terms. Consider again the demand function of a positive contributor, $\hat{Q}_i(\phi_i) = \hat{Q}_i(m_i + Q_{-i})$. The response of demand to a change in money income is the derivative $\hat{Q}_i'(.)$, and this may be termed the player's **marginal propensity to contribute** to the public good.

Now consider a change in the contributions of others. Observe that, if player i is a positive contributor, her preferred quantity of the public good may be written either as $\hat{Q}_i(\phi_i)$ or, using the replacment function, as $Q_{-i} + r_i(Q; m_i)$. At an equilibrium, therefore, we may write

$$\hat{Q}_i(m_i + Q_{-i}) = Q_{-i} + r_i\left(\hat{Q}_i(m_i + Q_{-i}); m_i\right). \tag{9}$$

Now consider a change in Q_{-i}, and suppose that i is a positive conributor both before and after the change. Differentiating (9),

$$\hat{Q}_i'(.) = 1 + r_{iQ}(.)\hat{Q}_i'(.) \tag{10}$$

where $r_{iQ}(\cdot) = \frac{\partial r_i(\cdot)}{\partial Q}$. Rearranging,

$$r_{iQ}(\cdot) = \frac{\hat{\phi}_i'(\cdot) - 1}{\hat{\phi}_i'(\cdot)}. \qquad (11)$$

This result provides a useful link with the more familiar marginal propensity to contribute, which we exploit below. It also indicates the range of values that may be taken by the response $r_{iQ}(\cdot)$. Normality implies that $0 < \hat{\phi}_i'(\cdot) < 1$. Consequently, (11) implies that $-1 < r_{iQ}(\cdot) < 0$.

2.4.2 Comparative statics of equilibrium provision

The comparative static properties of the individual replacement functions lead directly to a number of interesting comparative static properties of equilibrium provision. The first is the celebrated neutrality property:

Corollary 1 (The Neutrality Property) Any redistribution of income between a set of positive contributors that leaves that set unchanged also leaves the equilibrium allocation unchanged.

Proof. Consider a Nash equilibrium at which $Q = Q^*$. Partition players into two sets: contributors [C] and noncontributors [NC]. Now consider a vector of changes in the incomes of contributors such that (i) $\sum_{j \in C} \Delta m_j = 0$ and (ii) at the initial equilibrium, the set of contributors is unchanged. Then, at the initial value of Q,

$$\Delta R(m : Q^*) = \sum_{j \in C} \Delta r_j(m_j : Q^*) = \sum_{j \in C} \Delta m_j = 0$$

Hence, it remains true that, at the initial value Q^*, $R(Q^*; m) = Q^*$. ■

The neutrality property implies that, in a well-defined sense, the set of positive contributors who face the same unit cost of public good provision behaves like a single individual. If attention is confined to income distributions that are consistent with a given set of positive contributors, then the aggregate replacement function associated with that set depends upon just two arguments: the total income of all contributors, and the value of Q.

Corollary 2 For all income distributions consistent with a given set C of players being the positive contributors to the public good in equilibrium, $R(\cdot) = R(Q; M_C)$, where $M_C = \sum_{j \in C} m_j$.

Consider the behaviour of the implied 'single contributor'. Specifically, consider the response of total equilibrium public good provision to a change in the total income received by the set of contributors. We assume throughout that the set of contributors is unchanged. At a Nash equilibrium, each contributor implicitly chooses the same total, so that $Q_j = Q$ for all $j \in C$, where Q_j is contributor j's most preferred total level of public good. Equilibrium is characterised by the following equation:

$$R(Q) = \sum_{j \in C} r_j(Q; m_j) = Q. \tag{12}$$

Now suppose that the contributors' income levels change. At the new equilibrium, it remains the case that the sum of replacement values equals the total provision. Differentiating (12),

$$\sum_{j \in C} r_{jQ}(\cdot)\, dQ + \sum_{j \in C} r_{jm}(\cdot)\, dm_j = dQ \tag{13}$$

However, we have already shown that $r_{jm}(\cdot) = 1$ for all $j \in C$. Writing $M_C \equiv \sum_{j \in C} m_j$, equation (13) becomes

$$\sum_{j \in C} r_{jQ}(\cdot)\, dQ + dM_C = dQ$$

or

$$\frac{dQ}{dM_C} = \frac{1}{1 - \sum_{j \in C} r_{jQ}(\cdot)}. \tag{14}$$

Substituting from 11,

$$\frac{dQ^N}{dM_C} = \frac{1}{1 + \sum_{j \in C} \left(\frac{1}{\hat{Q}'_j} - 1 \right)} = \frac{1}{1 - n_C + \sum_{j \in C} \left(\frac{1}{\hat{Q}'_j} \right)}.$$

To summarise,:

Proposition 2.8 *Let the aggregate income of the set of contributors change by an amount dM_C, and assume that the set of positive contributors is unchanged. Then the response of the equilibrium provision of the public good is given by*

$$\frac{dQ^N}{dM_C} = \frac{1}{1 - n_C + \sum_{j \in C} \left(1/\hat{Q}'_j \right)}. \tag{15}$$

This is precisely the result obtained by Cornes and Sandler (2000). To get a feeling for the magnitude of this response, suppose that contributors are identical, with $\phi'_j = \phi'$ for all $j \in C$. Then(15) may be written as

$$\frac{dQ^N}{dM_C} = \frac{\left(\phi'\right)}{\phi' + \left(1 - \phi'\right) n}. \tag{16}$$

Normality implies that $0 < \left(1 - \phi'\right) < 1$. If we slightly strengthen this condition by supposing that $\left(1 - \phi'\right)$ is bounded away from zero - that is, $\left(1 - \phi'\right) \leq \epsilon < 0$ - then (16) implies that

$$\lim_{n \to \infty} \frac{dQ^N}{dM_C} = 0.$$

For example, suppose that each has a constant marginal propensity to contribute of $\phi' = 0.5$. Then inspection of (16) shows that the aggregate equilibrium response in an n-player game is $dQ^N/dM_C = 1/\left(\left(\frac{1}{\phi'} - 1\right) n_C + 1\right) = 1/(n_C + 1)$. If $n = 10$, $dQ^N/dM_C = 1/11$. If $n_C = 100$, $dQ^N/dM = 1/101$. For a given common value of the individual marginal propensity, the magnitude of the aggregate propensity falls rapidly as n_C increases.

One further implication of (15) is worth noting. Consider an equilibrium in which the existing contributors are not identical. Denote by ϕ'_{min} the lowest of the individual marginal propensities to contribute, and suppose for simplicity that there is just one player whose marginal propensity to contribute takes this value. Then

$$\frac{dQ^N}{dM} = \frac{1}{1 + \sum_j^n \left(\frac{1}{\phi'_j} - 1\right)} = \frac{1}{\frac{1}{\phi'_{min}} + \sum_{\substack{j=1 \\ j \neq min}}^n \left(\frac{1}{\phi'_j} - 1\right)}. \tag{17}$$

Normality implies that each player's marginal propensity to contribute is less than one, which in turn implies that the summation term must be positive. (17) implies that, at any equilibrium allocation, the aggregate response dQ/dM is less than the smallest individual response ϕ'_{min}. Not only does the interaction between players' responses dampen the response of aggregate provision to any change in the income of the set of positive contributors. In addition, the presence of just one contributor with a low propensity to contribute is enough to place a precise upper bound on the aggregate propensity to contribute of a given set of positive contributors.

This suggests that a public good economy with a large number of potential contributors displays an 'approximate neutrality property' in the following sense. Suppose that players differ from one another with respect to both preferences and income. One can imagine drawing individuals from a joint distribution of preferences and income. For any given individual, specification of the preference map and income level determines that player's dropout value. Now order the individuals according to their dropout values. Denote by type-H the type with the highest dropout value. Suppose that there is a finite number of types. As more individuals are drawn from the joint probability distribution, the number of type-H's increases. There will be a number of type-H's, say n_H^* - such that, for any $n_H = n_H^*$, the resulting equilibrium involves only this type making a positive contribution. Although, as Andreoni [1] shows, the proportion of the total population that makes a positive contribution may be vanishingly small, it may yet be the case that the absolute number of type-H's - each of whom is a positive contributors - is large. If it is large enough, the aggregate marginal propensity to contribute may be vanishingly small. If this is the case, then redistribution amongst any types will have little effect on the equilibrium aggregate level of provision.

What are the normative implications of a redistribution of initial incomes in this model? We have shown that redistributions of initial income among positive contributors change nothing. Redistributions among noncontributors benefit the recipients and hurt the donors, leaving the utilities of all others unchanged. But what about redistributions from noncontributors to contributors? Cornes and Sandler [13] have recently shown that, even when every individual faces the same unit cost of contribution to the public good, such transfers can lead to a new Nash equilibrium that Pareto-dominates the equilibrium associated with the initial income distribution. This is easily shown in a simple two-type economy. Consider an equilibrium of a public goods economy at which there are n_N noncontributors and n_C positive contributors. The utility of a typical noncontributor is

$$u_N = u_N(y_N; Q) = u_N(m_N; Q)$$

Now suppose that the same amount of income is taken from each noncontributor and given to a positive contributor. To keep the exposition simple, assume the set of contributors is unchanged at the new equilibrium. Let the total extra income received by all contributors be dM_C. Each noncontributor loses an amount of income $dm_N = -dM_C/n_N$.

The change in utility of a typical noncontributor is

$$\begin{aligned} du_N &= \frac{\partial u_N(m_N, Q)}{\partial y_N} dm_N + \frac{\partial u_N(m_N, Q)}{\partial Q} dQ \\ &= \frac{\partial u_N(m_N, Q)}{\partial y_N} [\nu_N dQ + dm_N] \\ &= \frac{\partial u_N(m_N, Q)}{\partial y_N} [\nu_N dQ - dM_C/n_N] \end{aligned}$$

where $\nu_N \equiv \frac{\partial u_N(m_N, Q)/\partial Q}{\partial u_N(m_N, Q)/\partial y_N}$ is the noncontributor's marginal valuation of the public good. The fact that an individual is choosing not to contribute implies that, at the equilibrium, $\nu_N < c$. However, this is consistent with noncontributors placing a strictly positive valuation on the public good. The typical noncontributor will be better off if, in the course of adjustment to the new equilibrium, $\nu_N dQ - dM_C/n_N > 0$.

To determine whether noncontributors are made better off, we need to determine the endogenous response of total provision. We already know that

$$dQ = \left\{ \frac{\phi'}{\phi' + \left(1 - \phi'\right) n_C} \right\} dM_C$$

where ϕ' is the marginal propensity to contribute of the typical contributor. We also know that $dM_C = -n_N dm_N$. Substituting into (18),

$$du_N = \frac{\partial u_N(m_N, Q)}{\partial y_N} \left\{ \nu_N \left(\frac{\phi'}{\phi' + \left(1 - \phi'\right) n_C} \right) - 1/n_N \right\} dM_C$$

The right hand side is positive if the expression in square brackets is positive - that is, if

$$\frac{n_N \phi' \nu_N - \phi' - \left(1 - \phi'\right) n_C}{\left(\phi' + \left(1 - \phi'\right) n_C\right) n_C} > 0.$$

The denominator of this expression is positive. Therefore the utility of a noncontributor rises if the numerator is positive. Rearranging, this requires that

$$\widehat{Q}' > \frac{n_C}{n_N \sigma_N + n_C - 1}:$$

This makes sense. If n_N is large, each noncontributor is only one of many who are giving up income, and the gain in the aggregate income of contributors may be significant by comparison. Furthermore, the greater is $\widehat{Q}'$, the greater is the additional public good provision purchased by a given transfer of income.

3 Varying Unit Costs across Contributors

There has recently been a lot of interest in applying the public good framework to settings in which it is natural to allow unit costs to differ across potential contributors - see, for example, Arce and Sandler[3] and Sandler[21]. Ihori [17] applies the best response function approach to analyse such a model. The replacement function approach can accommodate this extension very easily. Throughout his section we allow the unit cost parameter c_i to vary across contributors and write player i's budget constraint as

$$y_i + c_i q_i \leq m_i:$$

We leave the reader to confirm that, for a given vector of initial incomes, this slight modification does not affect the listed properties of $r_i(Q)$. Consequently, the properties of the aggregate replacement function are unaffected, and the argument that establishes existence and uniqueness of a Nash equilibrium goes through exactly as before. This extension does not complicate subsequent comparative static analysis. However, it does change some of our conclusions in interesting ways. To begin with, the neutrality property associated with income redistribution no longer holds. However, before looking at the effects of income changes in this model, we first draw attention to a striking distributional implication of unit cost differences across contributors.

Proposition 3.1 Let players i and j be positive contributors in equilibrium. Suppose further that they have identical preferences and identical incomes, but differ with respect to their unit costs as contributors. Then at equilibrium, the contributor with higher unit cost enjoys a higher level of utility.

Proof. Let $c_i > c_j$. At the equilibrium level of public good provision Q^*, all contributors equate their marginal rates of substitution to their marginal costs of public good povision. Therefore

$$c_i > c_j \implies \frac{\partial u(y_i^*;Q^*)/\partial Q}{\partial u(y_i^*;Q^*)/\partial y_i} > \frac{\partial u\left(y_j^*;Q^*\right)/\partial Q}{\partial u\left(y_j^*;Q^*\right)/\partial y_j}.$$

Since both enjoy the same level of public good provision, normality implies that $y_i > y_j$. But, since preferences are identical, this implies that $u(y_i^*;Q^*) > u\left(y_j^*;Q^*\right)$. ■

In short, higher cost contributors are better off than otherwise identical lower cost contributors. It does not necessarily pay to be a low cost producer of the public good.

3.1 Comparative statics of income changes

Denote by the set C the set of positive contributors at the Nash equilibrium of a public good economy. A 'pure' redistribution amongst the contributors is a set of possibly discrete income changes Δm_i such that $\sum_{j\in C} \Delta m_j = 0$.

Proposition 2.6 is modified as follows:

Proposition 3.2 Let player i be a positive contributor both before and after an exogenous change in money income of Δm_i, and suppose that her unit cost is c_i. Then, if Q is held fixed, her replacement value changes by the amount $\Delta r_i(m_i : Q) = \Delta m_i / c_i$.

Starting at a Nash equilibrium, consider the effect on equilibrium of a pure redistribution among contributors. Assume that the set of positive contributors is not changed by the redistribution. At the initial equilibrium provision level, Q^*, the value of the aggregate replacement function rises, stays unchanged, or falls according to whether

$$\left(\sum_{j\in C} \Delta r_j = \right) \sum_{j\in C} \frac{\Delta m_j}{c_j} >; = \text{ or } < 0.$$

For a given set of incomes, the aggregate replacement function is nonincreasing in Q. Therefore the following corollary of proposition 3.2 holds.

Corollary 3 If a pure redistribution amongst the set of positive contributors leaves that set unchanged, then aggregate equilibrium provision rises, remains unchanged or falls according to whether $\sum_{j\in C} \frac{\Delta m_j}{c_j} >; = \text{ or } < 0$.

For example, redistribution from contributor A to contributor B increases equilibrium provision if A's unit cost exceeds B's. Redistribution from a higher to a lower cost contributor enhances efficiency, and the efficiency gain is partly taken through an increase in the provision of the public good.

Not only does such a redistribution increase equilibrium public good provision - it is also Pareto-improving. The reasoning is simple. Each individual's preference map in $(y;Q)$ space is fixed throughout the present thought experiment. Under the normality assumption, if each individual is enjoying a higher level of total public good provision, she must have moved upwards and to the right along her income expansion path. Hence, her consumption of the private good is higher, and so must be her utility. In short,

Corollary 4 If a pure redistribution amongst the set of positive contributors leaves that set unchanged, the new equilibrium is Pareto superior to, identical with, or Pareto inferior to the initial equilibrium according to whether $\sum_{j \in C} \frac{\Delta m_j}{c_j} >; =$ or < 0.

Of course, we need not confine attention to pure redistributions. A further implication of the present analysis is that, if unit costs differ across individuals, a reduction in the aggregate income of contributors is consistent with Pareto improvement. This follows from the simple observation that the inequalities $\sum_{j \in C} \Delta m_j < 0$ [a reduction in the aggregate income of contributors] and $\sum_{j \in C} \frac{\Delta m_j}{c_j} > 0$ [a Pareto improving change in contributors' incomes] are perfectly consistent with one another if unit costs vary across individuals.

3.2 Comparative statics of unit cost changes

Once we admit the possibility of varying unit cost levels across contributors, which invites a production theoretic interpretation of the model, it is natural to consider the implications of changes in the unit costs of contributors, reflecting a change in the technology available to them. Write player i's utility function as

$$u^i(y_i;Q) = u^i(m_i - c_i q_i;Q) = u^i\left(m_i - c_i r^i(Q;c_i);Q\right)$$

Now consider the effect of a change $(dc_1, dc_2, \ldots, dc_n)$ in the vector of unit costs:

$$\begin{aligned} du^i &= u^i_y \left[-r^i dc_i - c_i \left(r^i_Q dQ + r^i_c dc_i \right) + \pi_i dQ \right] \\ &= u^i_y \left[-\left(r^i + c_i r^i_c \right) dc_i + c_i \left(1 - r^i_Q \right) dQ \right] \end{aligned} \tag{18}$$

where we have used the fact that, if player i is a positive contributor, $\pi_i = c_i$. First, we must solve for dQ. Recall the characterization of equilibrium in terms of replacement functions:

$$\sum_j r^j(Q; c_j) = Q \;\rightarrow\; \sum_j \left(r^j_Q dQ + r^j_c dc_j \right) = dQ$$

$$\Rightarrow \; dQ = \frac{\sum_j \left(r^j_c dc_j \right)}{\left[1 - \sum_j r^j_Q \right]}$$

Substituting into (18)

$$du^i = u^i_y \left\{ -r^i dc_i - c_i \left(r^i_c dc_i \right) + c_i \left(1 - r^i_Q \right) \frac{\sum_j \left(r^j_c dc_j \right)}{\left[1 - \sum_j r^j_Q \right]} \right\}$$

$$du^i = u^i_y \left\{ -\left(r^i + c_i r^i_c - \frac{c_i r^i_c \left(1 - r^i_Q \right)}{\left[1 - \sum_j r^j_Q \right]} \right) dc_i + c_i \left(1 - r^i_Q \right) \frac{\sum_{j \neq i} \left(r^j_c dc_j \right)}{\left[1 - \sum_j r^j_Q \right]} \right\} \tag{19}$$

Consider some particular cases:

A change in c_i alone

$$du^i = -u^i_y \left\{ \left(r^i + \frac{c_i r^i_c \left[1 - \sum_{j \neq i} r^j_Q \right]}{\left[1 - \sum_j r^j_Q \right]} \right) dc_i \right\}$$

Normality $\Rightarrow$ $r^i_c < 0$. Thus the sign pattern is

$$du = (-) \{ [(+) + (-)] \, dc_i \}$$

which is ambiguous. Thus we have

Proposition 3.3 A reduction in the unit cost of contributor i may either raise or reduce her equilibrium level of utility.

In the symmetric Cobb-Douglas example, the indirect effect through the response of Q, dominates, and an increase in a player's unit cost increases her equilibrium utility level.

A change in c_k alone, where $k \neq i$ Interestingly, a reduction in contributor k's unit cost has an unambiguously beneficial effect on i's utility. Equation ?? becomes

$$du^i = u^i_y \left\{ c_i \left(1 - r^i_Q\right) \left[\frac{r^k_c dc_k}{1 - \sum_j r^j_Q}\right] \right\}$$

A reduction in k's unit cost means $dc_k < 0$. Furthermore, $r^k_c < 0$. Hence $du^i > 0$.

Proposition 3.4 A reduction in the unit cost of contributor k will raise the equilibrium level of utility of all other contributors [and of noncontributors].

An equal proportional change in all unit cost levels. Write player i's unit cost as λc_i. Then an equal proportional change in each player's unit cost is modelled as a change in λ.

$$du^i = u^i_y \left\{ - \left(r^i + c_i r^i_c - \frac{c_i r^i_c \left(1 - r^i_Q\right)}{1 - \sum_j r^j_Q} \right) c_i + c_i \left(1 - r^i_Q\right) \left[\frac{\sum_{j \neq i} (r^j_c c_j)}{1 - \sum_j r^j_Q}\right] \right\} d\lambda$$

$$du^i = -u^i_y \left\{ \left[r^i + r^i_c c_i\right] - \frac{\left(1 - r^i_Q\right) \sum_j (r^j_c c_j)}{1 - \sum_j r^j_Q} \right\} c_i d\lambda$$

$$du^i = -u^i_y \left\{ r^i + r^i_c c_i \left\{ 1 - \left[\frac{\left(1 - r^i_Q\right)}{1 - \sum_j r^j_Q}\right] \right\} - \frac{\left(1 - r^i_Q\right) \sum_{j \neq i} (r^j_c c_j)}{1 - \sum_j r^j_Q} \right\} c_i d\lambda$$

The sign pattern is:

$$du^i = -(+)\left\{(+) + (-) - (-)\right\}(+)\, d\theta$$

The expression cannot be signed.

Proposition 3.5 An equal proportional reduction in all contributors' unit costs may or may not raise the equilibrium level of utility enjoyed by contributor i.

However, if players are identical, so that the game is symmetric, then we can obtain a stronger result. The change in utility of the representative contributor is

$$\begin{aligned} du &= -u_y \left\{ r + \frac{r_c c[1 - n r_Q] - (1 - r_Q) n r_c c}{[1 - n r_Q]} \right\} c_i d\theta \\ &= -u_y \left\{ r + \frac{r_c c(1 - n)}{[1 - n r_Q]} \right\} c_i d\theta \end{aligned}$$

The sign pattern is

$$du = -(+)\left\{(+) + (+)\right\}(+)\, d\theta$$

Thus, $du/d\theta < 0$: an equal proportional increase in each player's unit cost reduces the utility of each.

Proposition 3.6 If all contributors are identical, an equal proportional reduction in every contributor's unit cost raises the level of utility enjoyed by each at the symmetric equilibrium.

This property of the symmetric contribution game was obtained by Cornes and Sandler [13].

Example 5 We finish this section with a specific example which illustrates clearly some of the comparative static possibilities opened up by allowing parametric variation in unit cost levels. Let each of n players have the Cobb-Douglas utility function $u(y; Q) = yQ$ and let the budget constraint of i be $y_i + c_i q_i = 1$.

Player i's replacment function is readily shown to be

$$q_i = r_i(Q : c_i) = \max\left\{\frac{1}{c_i} - Q; 0\right\}$$

Assuming there are n contributors, the equilibrium level of provision is

$$Q = \frac{1}{1+n}\sum_{j=1}\left(\frac{1}{c_j}\right)$$

and the equilibrium level of utility enjoyed by contributor i is

$$\begin{aligned} u_i &= (1 - c_i q_i)\,Q = \left\{1 - c_i\left(\frac{1}{c_i} - Q\right)\right\}Q = c_i Q^2 \\ &= c_i\left\{\frac{1}{1+n}\sum_{j=1}\left(\frac{1}{c_j}\right)\right\}^2 \end{aligned}$$

To begin, suppose that there are two contributors. Let the unit cost of contributor 2 be unity, and consider how contributor 1's equilibrium utility varies with her unit cost. Substituting the appropriate value of n and c_2, we get

$$u_1 = \frac{1}{9}\frac{(c_1+1)^2}{c_1}$$

Figure 4 graphs this function. For all $c_1 < 0.5$; contributor 1 is the sole contributor. For $0.5 < c_1 < 2$, all contribute. If $c_1 \geq 2$, contributor 1 will not find it worthwhile to contribute in equilibrium. Observe that, up to the point where the unit costs of the individuals are equal, an increase in 1's unit cost reduces her equilibrium utility. Thereafter, further increases in her unit cost benefit her. The symmetric situation, in which, $c_1 = c_2$, is precisely the point at which the effect of contributor 1's cost on her own equilibrium utility is of second order.

If the example is now extended to allow for more than two players, the picture changes slightly. Suppose there are 4 players, and that $c_2 = c_3 = c_4 = 1$. Figure 5 shows how player 1's equilibrium utility varies with her unit cost. For sufficiently low values of c_1, player 1 is the sole contributor, and increases in her unit cost lower her equilibrium utility. At $c_1 = 0.5$, the other players enter, and from that point onwards, increases in c_1 have a first-order negative effect on her equilibrium utility. At $c_1 = 4/3$, contributor 1 drops out, and from that point onwards further increases in c_1 have no effect on her equilibrium utility. It is striking that, throughout the range of values for which all contribute, the effect of changes in c_1 on the equilibrium level of u_1 is 'perverse'. This has interesting implications for any extension of this model that allows players to take steps to enhance, or lower, their productivity as contributors prior to playing the contribution game.

4 A More General Social Composition Function

In the basic public good model, the aggregate quantity of the public good is simply the unweighted sum of individual contributions. We now examine a broader class of social composition functions, in which Q is an additively separable function of individual contributions. Specifically, we maintain the following assumption in all that follows:

A.6: Additively separable social composition function

$$Q = G\left\{\sum_{j=1} f_j(q_j)\right\} \tag{20}$$

> where the functions $G[\cdot]$ and $f_j(\cdot)$ are everywhere strictly increasing and twice continuously differentiable.

This class includes not only the basic model, obtained by putting $G[\sigma] = \sigma$ and $f_j(q_j) = q_j$, and weighted sum models in which $f_j(q_j) = b_j q_j$, but also more general production processes through which the inputs - or individual contributions - generate the output Q.

We now show that a public good game in which the social composition function satisfies A.6 may be transformed into an aggregative game by a simple change of variables. Define $z_i = f_i(q_i)$, $i = 1,...,n$, and $Z = \sum_{j=1}^{n} z_j$, so that $Q = G(Z)$. Our assumptions imply that these functions have inverses, and we will write $q_i = f_i^{-1}(z_i) = g_i(z_i)$ and $Z = G^{-1}(Q) = F(Q)$. Player i's payoff function may be written as

$$\begin{aligned} u_i(y_i;Q) &= u_i\left(m_i - q_i; G\left\{\sum_{j=1} f_j(q_j)\right\}\right) \\ &= u_i\left(m_i - g_i(z_i); G\left\{\sum_{j=1} z_j\right\}\right) \\ &= \pi_i(z_i;Z). \end{aligned}$$

This justifies the following claim:

Claim 6 *If the social composition function $\Phi(\cdot)$ satisfies A.6, the generalized pure public good model may be transformed into an aggregative game.*

Our transformation has converted the game into one in which player i's problem is

$$\underset{y_i, z_i}{\text{Maximise}} \{u_i(y_i, z_i + Z_{-i}) \mid y_i + g_i(z_i) \leqq m_i\}.$$

Player i's marginal rates of substitution and transformation can be expressed as functions of the transformed variables:

$$\frac{\partial u_i(y_i, Q)/\partial y_i}{\partial u_i(y_i, Q)/\partial Q} = MRS_i(m_i - q_i, Q) \tag{21}$$

$$= MRS_i(m_i - g_i(\sigma_i Z), G(Z)) = \zeta_i(\sigma_i, Z), \text{ say,} \tag{22}$$

and

$$MRT_i = G'\left\{\sum_j f_j(q_j)\right\} f_i'(q_i) = G'[Z]\, f_i'[g_i(\sigma_i Z)] = \tau_i(\sigma_i, Z). \tag{23}$$

Note that we define player i's marginal rates of substitution and of transformation as functions, not of her contribution q_i and Q, but of her share $\sigma_i \equiv q_i/Q$ and Q. The first order conditions may be written as

$$\zeta_i(\sigma_i, Z) \geqq \tau_i(\sigma_i, Z) \tag{24}$$

$$[\zeta_i(\sigma_i, Z) - \tau_i(\sigma_i, Z)]\, g_i(\sigma_i Z) = 0. \tag{25}$$

We analyze player i's behaviour by examining the first order conditions (24) and (25). To obtain further results, we must make further assumptions about the form of the social composition function $\Phi(q)$. Throughout the rest of this section, we adopt the following assumption:

- PG.2. $\Phi(q)$ takes the form:

$$\Phi(q) = \left(\sum_{j=1} b_j q_j^{\alpha}\right)^{\gamma/\alpha}; \; b_j > 0, \alpha \leqq 1, 0 < \gamma \leqq 1. \tag{26}$$

This is a standard CES production function embodying nonincreasing returns to scale - indeed, a convex technology. Inspection of (21) reveals that, for any given value of Z - and therefore of Q - an increase in σ_i implies a reduction in y_i and therefore, in view of the normality assumption, an increase in the value of $\zeta_i(\cdot)$. This is reflected in the shape of the graph of $\zeta_i(\cdot)$ in Figure 6. Now consider player i's marginal rate of transformation. From (23), PG.2 and the fact that $z_j = b_j q_j^{\alpha}$

$$\begin{aligned} \tau_i(\sigma_i; Z) &= G'[Z] f_i'[g_i(\sigma_i Z)] = \gamma Z^{\gamma - 1} \alpha_j b_j q_j^{\alpha - 1} \\ &= \gamma [Z]^{\gamma - 1} \alpha_j b_j^{\frac{1}{\alpha}} \sigma_j^{\frac{\alpha - 1}{\alpha}} Z^{\frac{\alpha - 1}{\alpha}} = k_j \sigma_j^{\frac{\alpha - 1}{\alpha}} Z^{\frac{\alpha\gamma - 1}{\alpha}} \end{aligned} \tag{27}$$

where $k_j = \gamma \alpha_j b_j^{\frac{1}{\alpha}} > 0$: Clearly, since by assumption $\alpha \leqslant 1$, $\tau_i(\sigma_i; Z)$ is nonincreasing in σ_i. Again, this is shown in Figure 6[1]. This establishes the following result:If PG.2 is satisfied, every player has a well-defined share

function $s_i(Z)$.

Now consider how σ_i varies in response to changes in Z. An increase in Z, with σ_i held constant, is associated with an increase in Q and a reduction in $y_i = m_i - g_i(\sigma_i Z)$. On both counts, it increases the value of $\zeta_i(\cdot)$ associated with any given value of σ_i. Thus an increase in Z shifts the graph of $\zeta_i(\cdot)$ upwards. At the same time, since $\alpha_j \gamma - 1 \leqslant 0$, (27) implies that an increase in Z reduces the value of $\tau_i(\cdot)$ associated with any given value of σ_i. It therefore shifts the graph of $\tau_i(\cdot)$ downwards. Figure 7 shows the positions of the graphs of $\zeta_i(\cdot)$ and $\tau_i(\cdot)$ associated with the values $Z^0$and $Z^1 > Z^0$. Consequently, if the initial share value is positive, an increase in Z implies a strict reduction in the share value:

Claim 7 If PG.2 is satisfied, then player i's share function $s_i(Z)$ is everywhere nonincreasing, and is strictly decreasing wherever $s_i(Z) > 0$.

This implies that the aggregate share function, $s^*(Z) \equiv \sum_{j=1}^{n} s_j(Z)$, is a monotonic decreasing function wherever $s^*(Z) > 0$.

The implied properties of the aggregate share function lead readily to the following observation:

Claim 8 If PG.2 is satisfied, then the generalized public good game possesses a unique Nash equilibrium .

[1] The possibility of $\sigma_i = 0$ arises when the graph of $\zeta_i(\cdot)$ lies everywhere above that of $\tau_i(\cdot)$ given the prevailing value of Z.

We finish this example with a brief analysis of the implications of income transfers in the present model. It is well-known that the conventional public good model, in which $\alpha_j = b_j = \gamma = 1$ for all j and consequently $Q = \sum_{j=1}^{n} q_j$, exhibits the neutrality property. A transfer of initial income from one positive contributor to another has no effect on the equilibrium resource allocation. We now show that, more generally, the neutrality property does not hold and we investigate the circumstances under which a transfer from a higher income to a lower income contributor increases the aggregate equilibrium provision of the public good. Our analysis generalizes the result of Cornes (1993), who demonstrates this possibility and the consequent possibility of a Pareto improving redistribution when the aggregator function is a symmetric Cobb-Douglas, obtained by letting $\alpha \to 0$.

At any given Nash equilibrium, a transfer of income from one player to another will increase equilibrium provision, leave it unchanged or reduce it according to whether the sum of the two players' share value rise, remain unchanged or fall at that unchanged provision level. Although we cannot solve explicitly for an individual's share function, we can write down an explicit function for its inverse, which we write as $m_i = \mu_i(\sigma_i; Z)$. The partial derivatives of this function with respect to σ_i provide the information required to determine the shape of this function. This in turn allows us to infer the consequences of income redistribution for total provision of the public good.

Let all individuals have the same Cobb-Douglas utility function, $u_i(y_i; Q) = y_i Q$. The public good aggregator function is given by (26) with all $b_i = 1$ and $a_i = a_j$ for all $i \neq j$. At equilibrium, the share value of a strictly positive contributor is determined implicitly by the requirement that $\zeta_i(\cdot) = \tau_i(\cdot)$. Dropping the 'i' subscript, the relationship between an individual's income and that individual's share value at a given equilibrium is:

$$\frac{Z^{\gamma/\alpha}}{m - z^{1/\alpha}} = \gamma\alpha Z^{(\gamma-\alpha)/\alpha} z^{(\alpha-1)/\alpha}$$

or

$$m = \mu(\sigma; Z) = \frac{Z^{1/\alpha}}{\alpha\gamma}\left[\sigma^{(1-\alpha)/\alpha} + \alpha\gamma\sigma^{1/\alpha}\right]. \qquad (28)$$

The partial derivatives of $\mu(\sigma; Z)$ with respect to σ are

$$\frac{\partial \pi}{\partial \sigma} = \frac{Z^{1/\alpha}}{\alpha^2 \gamma}\left((1-\alpha)\,\sigma^{(1-2\alpha)/\alpha} + \alpha\gamma\,\sigma^{(1-\alpha)/\alpha} \right) > 0 \text{ if } \sigma > 0, \tag{29}$$

$$\frac{\partial^2 \pi}{\partial \sigma^2} = \frac{(1-\alpha)\,Z^{1/\alpha}\sigma^{(1-3\alpha)/\alpha}}{\alpha^3 \gamma}\{1 - 2\alpha + \alpha\gamma\sigma\}. \tag{30}$$

The signs of these expressions depend on the assumed value of the parameter α. It is convenient to consider five cases, according to the value of α. If Z is held constant, and π is regarded as a function of σ, the following conclusions may be drawn from (29) and (30).

CASE I: $\alpha < 0$: $\pi(\sigma; Z)$ is (i) strictly decreasing and (ii) strictly concave.

A player's share is a strictly decreasing and strictly convex function of income. The graph of $\pi(\sigma : Z)$ is shown in Panel (a) of Figure 8. If an equalizing transfer is made from a higher to a lower income individual, the donor's share value will rise, but by less than the fall in that of the recipient. Thus the equilibrium level of Z will fall. Recall that if $\alpha < 0$, Z is a decreasing function of Q. Consequently, we can conclude

Claim 9 If $\alpha < 0$, then

1. an equalising redistribution from richer to poorer leads to a greater quantity of the public good, and
2. equalising incomes amongst a subset of players increases the quantity of public good.

CASE II: "$\alpha = 0$": We use inverted commas because the function $\phi(.)$ is not well-defined for $\alpha = 0$. However, by letting $\alpha \to 0$ from above, we generate the Cobb-Douglas form. This is the case analysed by Cornes (1993), who shows that an equalizing transfer from a higher to a lower income individual will lead to an increase in the equilibrium level of Q. Because this requires a different treatment, and has already been analysed, we do not explicitly treat this case here.

CASE III: $0 < \alpha \leq 1/2$: $\pi(\sigma : Z)$ is (i) strictly increasing and (ii) strictly convex.

In this, and in the remaining cases, Z is increasing in Q. Since the player's share is a strictly increasing and strictly concave function of income, a transfer from a high to a lower income individual will reduce the former's share value, but will increase that of the latter by more. This is clearly seen in Panel (b) of the figure. Therefore at the initial equilibrium value of Z the aggregate share will increase. Thus the equilibrium value of Z, and therefore of Q, increases. Again we can state

Claim 10 If $0 < \alpha \leqslant 1/2$, then

1. an equalising redistribution from richer to poorer leads to a greater quantity of the public good.
2. Equalising incomes amongst a subset of players increases the quantity of public good.

CASE IV: $1/2 < \alpha < 1$: $\mu(\sigma : Z)$ is (i) strictly increasing, (ii) strictly concave for $0 \leqslant \sigma < (2\alpha - 1)/\alpha\gamma$ and (iii) strictly convex for $(2\alpha - 1)/\alpha\gamma < \sigma \leqslant 1$. Furthermore, $\mu(\sigma; Z)$ has (iv) slope unbounded above as σ approaches 0, and (v) a point of inflection at $\sigma = (2\alpha - 1)/\alpha\gamma$.

Panel (c) shows the graph of $\mu(\sigma : Z)$. Since there is a critical value of income below which the share function is concave, and above which it is convex, a transfer from a higher to a lower income individual has an ambiguous effect on the equilibrium level of Q. The following claim can be made:

Claim 11 If $1/2 < \alpha < 1$, then

1. an equalising redistribution from richer to poorer when the richest person involved has income no greater than $m^*(Z)$, where Z is the equilibrium level of Z, leads to a smaller quantity of the public good, and
2. equalising incomes amongst a subset of players when the richest person involved has income no greater than $m^*(Z)$, where Z is the equilibrium level of Z, leads to a smaller quantity of the public good.

Note that if there is a sufficient number of players - say a finite set of types with many of each type or a large number of independent selections from a distribution over income - then all shares will be less than $(2\alpha - 1)/\alpha\gamma$, which means that all incomes are less than $m^*(Z)$. Consequently, we would expect that, in the presence of many players, equalizing transfers will reduce the equilibrium quantity of Q. The ambiguity of this result may seem anomalous, in view of the lack of ambiguity in cases I-III and the neutrality proposition associated with case V. However, further consideration of case V explains the apparent anomaly.

CASE V: ¹ (¾ : Z) is (i) nondecreasing and (ii) piecewise linear.

This is the standard additive public good model. Transfers between positive contributors do not affect the equilibrium level of public good provision. However, for any given level of Z, there is a level of income $\underline{m}$ such that, for all $m < \underline{m}$, an individual will not contribute. Hence the piecewise linear nature of the relationship between ¾ and m. A transfer between a high income contributor and a low income noncontributor will reduce the share value of the former. However, the share value of the latter will generally either remain unchanged or increase by less than the fall in the share value of the donor. Therefore, the sum of the two players' share value falls.

5 Concluding Comments

Aggregative games can be analysed by conditioning the choices of individual players on the sufficient statistic that appears as an argument in the payoff of each. In contrast to the best response function, which conditions player i's choice on the choices of all players excluding player i, this simple trick avoids the unwelcome proliferation of dimensions as the number of heterogeneous players increases. It thereby permits a simple analysis of asymmetric aggregative games even with many players. It also lends itself to an elementary and revealing geometric representation.

We have confined attention to particularly well-behaved aggregative games. However, we have elsewhere analysed more complicated games - see Cornes and Hartley (2000, 2001) - and feel confident that the reach of our approach can extend further to analyse models for which the 'best response function' approach is not well suited. For this reason, we believe that replacement and share functions are tools that merit serious attention as the most natural tools for analysing aggregative games.

Our discussion of these functions and their application to public good and sharing games has hardly scratched the surface of a potentially significant range of applications. Aggregative games play a major role in many other economic models, of which Cournot oligopoly and contest theory are but two examples. The use of the replacement and share functions o¤ers the prospect of further insights in these and in many other applications.

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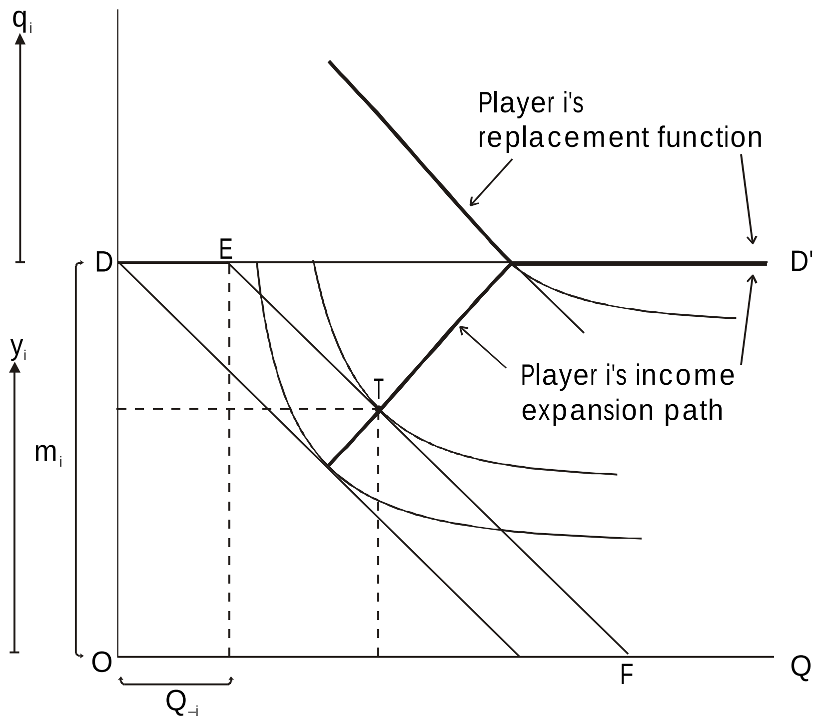
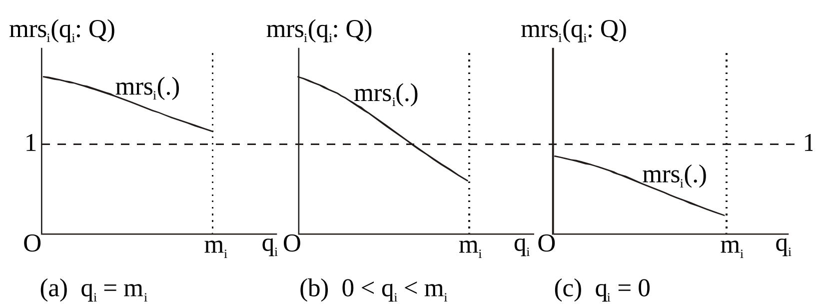


Figure 1:



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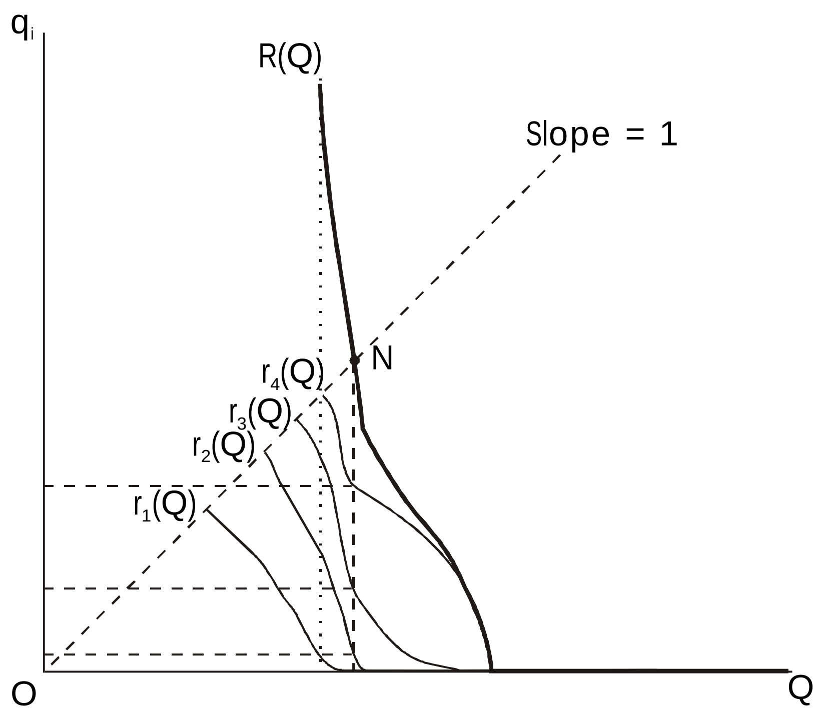

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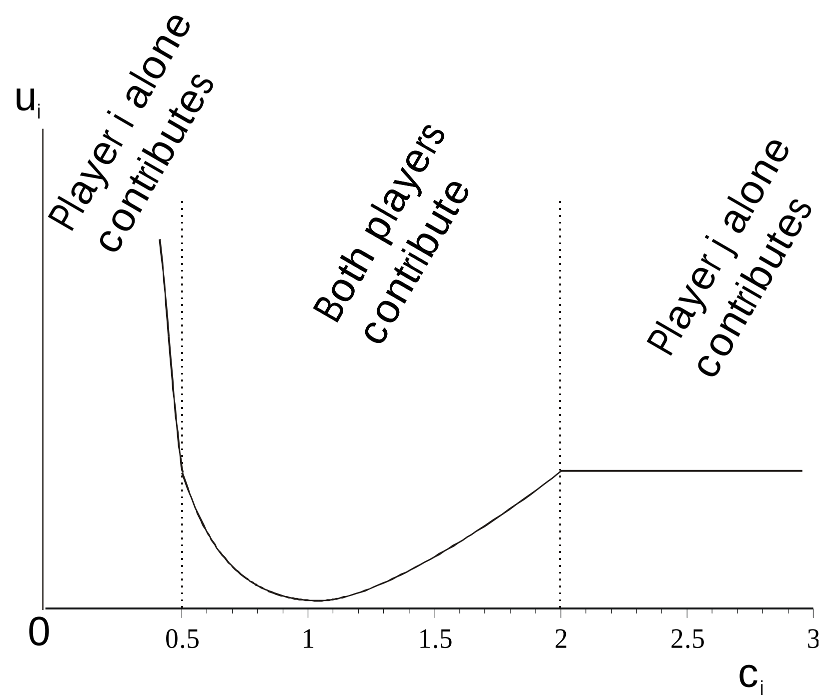
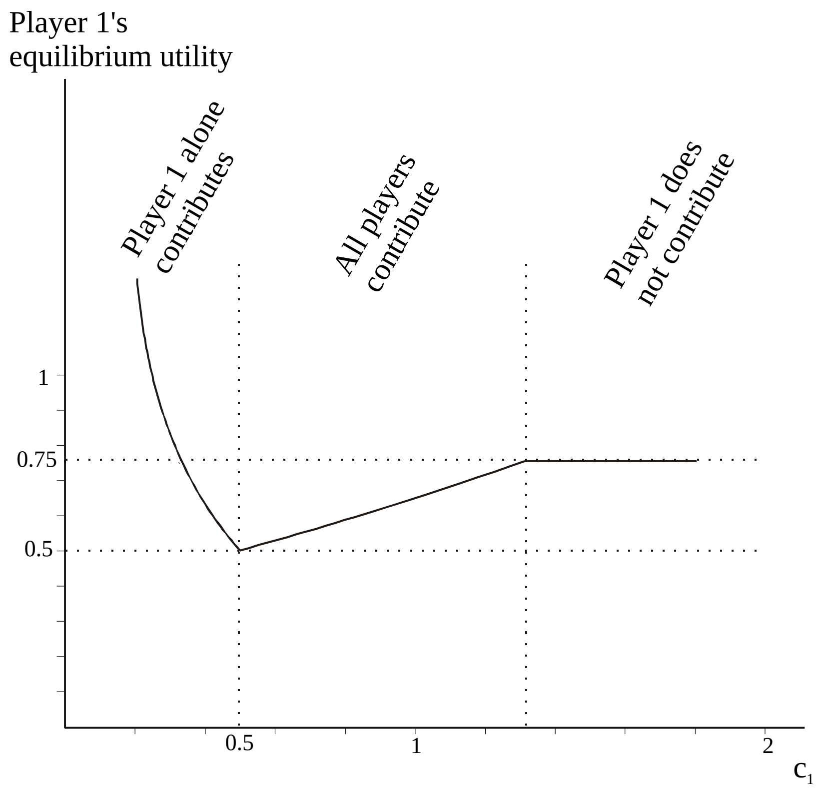

u_i
Player i alone contributes
Both players contribute
Player j alone contributes
0
0.5
1
1.5
2
2.5
3
c_i

Figure 4:



Player 1's
equilibrium utility
Player 1 alone
contributes
All players
contribute
Player 1 does
not contribute
1
0.75
0.5
0.5
1
2
c_1

Figure 5:

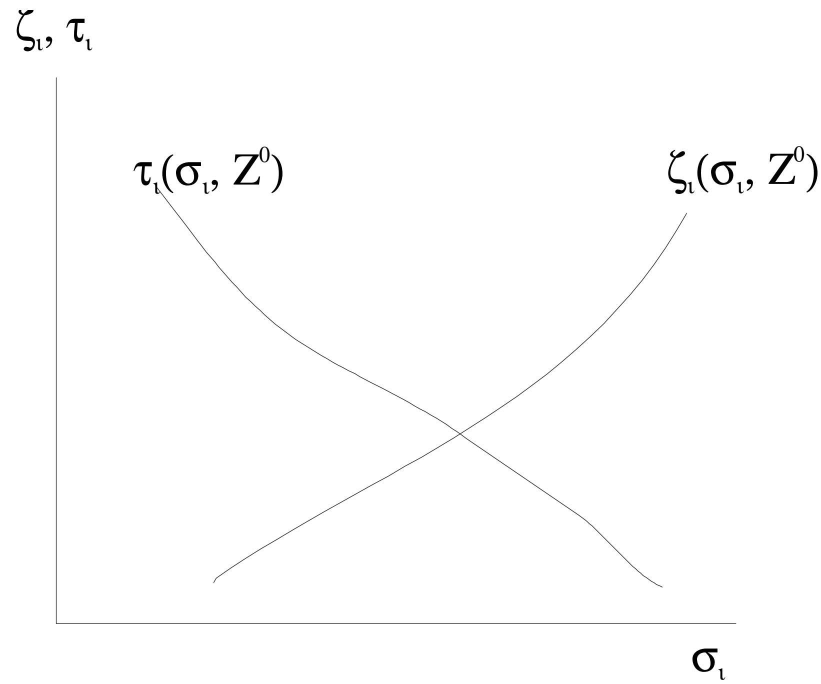
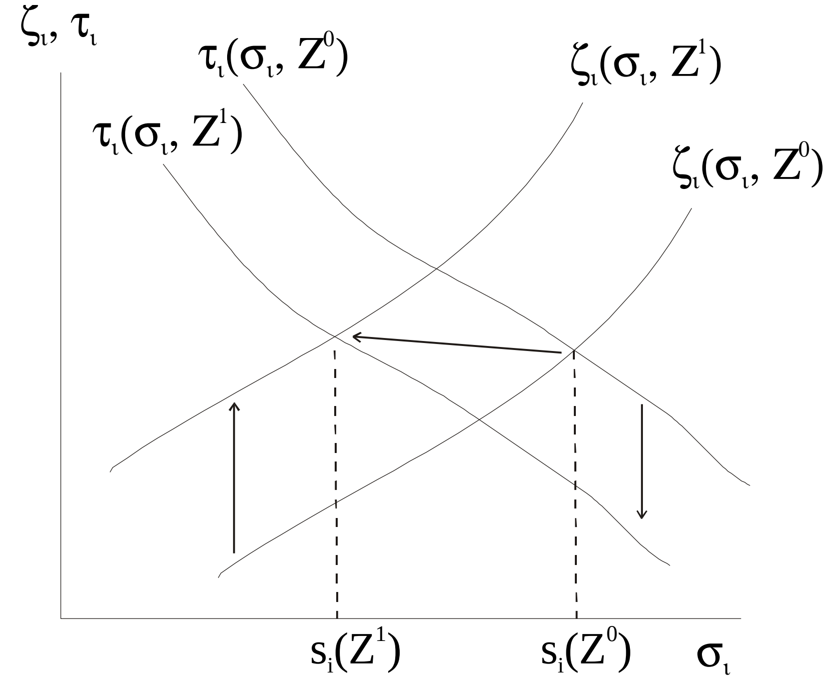

ζ_ι, τ_ι
$\tau_\iota(\sigma_\iota, Z^0)$
$\zeta_\iota(\sigma_\iota, Z^0)$
σ_ι

Figure 6:



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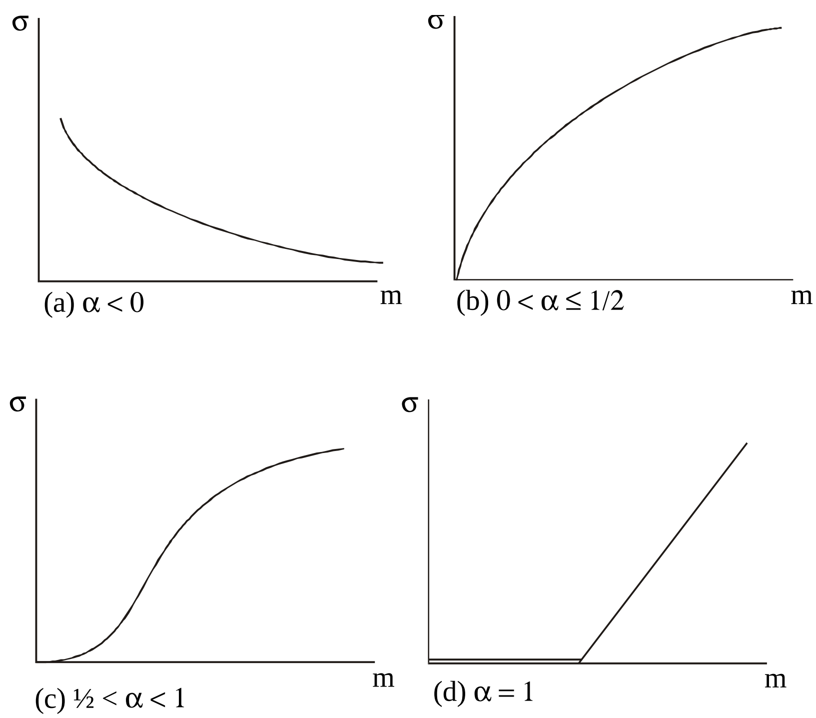

Figure 8: